Roll No. ☐☐☐☐☐☐☐☐☐☐☐☐ **Total No. of Pages : 02**

Total No. of Questions : 09

B.Tech. (AI & ML/DS)(CE)(CSE)(IT)(CSE and Design) (EE) (ECE) (EEE)(Robotics & Artificial Intelligence) (Internet of Things and Cyber Security including Block Chain Technology)(Block Chain)(ME)(FT)(Sem.–1)

ENGINEERING MATHEMATICS-I

Subject Code : BTAM101/23

M.Code : 93796

Date of Examination : 08-05-2024

Time : 3 Hrs. **Max. Marks : 60**

INSTRUCTIONS TO CANDIDATES :

1. **SECTION-A is COMPULSORY consisting of TEN questions carrying TWO marks each.**
2. **SECTION - B & C have FOUR questions each.**
3. **Attempt any FIVE questions from SECTION B & C carrying EIGHT marks each.**
4. **Select atleast TWO questions from SECTION - B & C.**

SECTION-A

1. Answer briefly :

a) What do you mean by convergent sequence?

b) Prove that the sequence $\left\{\dfrac{2n-7}{3n+2}\right\}$ is bounded.

c) Prove that $\sum\left(\dfrac{n}{n+1}\right)^2$ is divergent.

d) Find the length of the arc of the parabola $2y = x^2$ from $x = a$ to $x = b$.

e) Test for convergence of integral $\int_{e^2}^{\infty} \dfrac{dx}{x\log(\log x)}$.

f) Define Beta function.

g) Find first order partial derivative of $u = \tan^{-1}\dfrac{x^2+y^2}{x+y}$.

h) Show that the function $f(x,y)=\dfrac{xy^2}{x^2+y^4}$ has no limit as $(x, y) \to (0,0)$.

i) Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx.$

j) Evaluate $\int_0^2\int_1^2\int_0^{yz} x\ y\ z\ dx\ dy\ dz.$

SECTION-B

2. Prove that the sequence $\{a_n\}$ where $a_n = 1+\dfrac{1}{2}+\dfrac{1}{3}\ldots\ldots\dfrac{1}{n}-\log n$ is convergent.

3. Discuss the convergence or divergence of the series $\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{1}{n}.$

4. The curve $r = a(1+\cos\theta)$ revolves about the initial line. Find the volume of the figure formed.

5. Prove that $\beta(m,n)=\int_0^1 \dfrac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$ where $m > 0$, $n > 0$.

SECTION-C

6. If $u = x^3 + y^3 + x^3 + 3xyz,$ show that $x\dfrac{\partial V}{\partial x}+y\dfrac{\partial V}{\partial y}\ z\dfrac{\partial V}{\partial z}=3u.$

7. Obtain Taylor's expansion for $f(x,y) = y^x$ at (1, 1) up to second-degree term.

8. Evaluate $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$, by change of order of integration.

9. Find the volume of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

NOTE : Disclosure of Identity by writing Mobile No. or Making of passing request on any page of Answer Sheet will lead to UMC against the Student.

Re-apper exam

B.Tech (All Branches) ENGINEERING MATHEMATICS-1

Subject Code: BTAM 101/23, Mcode: 93796

Section-A (08-05-2024)

1. (a) What do you mean by Convergent Sequence?

Ans A sequence $\{a_n\}$ is said to converge to a limit l, if given $\epsilon > 0$, however small, there exist a positive integer m such that $|a_n - l| < \epsilon$ $\forall\ n \geq m$ and we can write it as

$$\lim_{n \to \infty} a_n = l \quad \text{or} \quad \text{lt}\ a_n = l.$$

(b) Prove that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is bounded.

Solⁿ:

$$a_n = \frac{2n-7}{3n+2}.$$

Put $n = 1, 2, 3$

$$a_1 = \frac{2(1)-7}{3(1)+2} = -1$$

$$a_2 = \frac{2(2)-7}{3(2)+2} = -3/8$$

$$a_3 = \frac{2(3)-7}{3(3)+2} = \frac{-1}{11}$$

$\therefore$ $a_n \geq -1$ $\forall n$

$\Rightarrow$ $\{a_n\}$ is bounded below.

Again $a_n = \dfrac{2n-7}{3n+2}$

$$3n+2 \,\big)\, 2n-7 \,\big(\, \tfrac{2}{3}$$
$$\underline{2n + 4/3}$$
$$-25/3$$

By quotient remainder rule

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \text{Remainder}$$

$$\frac{2n-7}{3n+2} = \frac{2}{3} - 25/3 < 2/3 \ \forall n$$

$\therefore$ $\{a_n\}$ is bounded above

$\Rightarrow$ $\{a_n\}$ is bounded.

(c) Prove that $\sum \left(\dfrac{n}{n+1}\right)^2$ is divergent.

Solⁿ:

$$a_n = \left(\frac{n}{n+1}\right)^2$$

$$a_n = \left(\frac{1}{1+1/n}\right)^2$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^2$$

$$= \left(\frac{1}{1+0}\right)^2 = 1$$

$$\therefore \lim_{n\to\infty} a_n \neq 0$$

$\Rightarrow$ $\sum a_n$ is divergent

Find length of the arc of parabola $x^2 = 2y$ from $x = a$ to $x = b$.

Solution :- Equation of parabola is

$$x^2 = 2y$$

Now $y = \frac{x^2}{2}$ —(1)

On differentiating eqn (1) with respect to x, we get

$$\frac{dy}{dx} = x$$

$$\text{Length of arc} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1+x^2}\, dx$$

$$= \left| \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\sin^{-1} x \right|_a^b \quad \left[\because \int \sqrt{a^2+x^2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left[\frac{b}{2}\sqrt{1+b^2} + \frac{1}{2}\sin^{-1} b\right] - \left[\frac{a}{2}\sqrt{1+a^2} + \frac{1}{2}\sin^{-1} a\right]$$

f) Define Beta function.

Ans:- Beta function is defined as,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx, \text{ where } m>0, n>0$$

Eg:- $\int_0^1 x^3(1-x)^5\, dx = \beta(4,6)$.

(e) Test convergence of integral $\int_{e^2}^{\infty} \frac{1}{x\log(\log x)}\,dx$

Sol:- Let $I = \int_{e^2}^{\infty} \frac{1}{x\log(\log x)}\,dx$

Put $\log x = t$ | when $x = e^2$, $t = \log e^2 = 2$

$\frac{1}{x}\,dx = dt$ | $x = e^2$; $t = 2$

when $x \to \infty$, $t \to \infty$

$$= \int_2^{\infty} \frac{1}{\log t}\,dt$$

Let $f(t) = \frac{1}{\log t}$, $g(t) = \frac{1}{t^m}$, $0 < m \le 1$

$$\therefore \frac{f(t)}{g(t)} = \frac{t^m}{\log t}$$

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{t^m}{\log t} \;\; \cancel{= mt^{m-1}} \;\; \left(\frac{\infty}{\infty} \text{ form}\right)$$

using L'Hospital Rule

$$\lim_{t\to\infty} \frac{mt^{m-1}}{1/t} = \lim_{t\to\infty} mt^m \to \infty$$

$\therefore$ by comparision test, if $g(t)$ is divergent then $f(t)$ is also divergent

As $\int_2^{\infty} \frac{1}{t^m}\,dt$ is divergent

$\therefore$ I diverges at ∞.

Find the first order partial derivative of
$u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Ans

$$\tan u = \frac{x^2+y^2}{x+y}$$

$$= \frac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{x\left[1+y/x\right]} = x^1 f(y/x) = Z \ (\text{say})$$

$\tan u = Z$ is a homogeneous function of degree 1

$\therefore$ By Euler Thm

$$x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = Z \quad (1)$$

Now $Z = \tan u$

$$\frac{\partial Z}{\partial x} = \sec^2 z \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial Z}{\partial y} = \sec^2 z \frac{\partial z}{\partial y}$$

So Put in (1)

$$x \sec^2 z \frac{\partial z}{\partial x} + y \sec^2 z \frac{\partial z}{\partial y} = Z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\tan z}{\sec^2 z}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\sin z}{\cos z} \times \cos^2 z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin z \cos z$$

(d) Show that the function $f(x,y) = \frac{xy^2}{x^2+y^4}$ has no limit as $(x,y) \to (0,0)$

Q'h'

given $f(x,y) = \frac{xy^2}{x^2+y^4}$

Let $(x,y) \to (0,0)$ along the Curve $x = my^2$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \frac{(my^2)y^2}{m^2y^4+y^4} = \frac{m}{m^2+1}$$

It is not unique as it takes different values for different values of 'm'

$\therefore f(x,y) = \frac{2x^2y}{x^4+y^2}$ has no limit as $(x,y) \to (0,0)$

(i) Evaluate: $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Ans(i) $I = \int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Integrating w.r.t 'y'

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(x^2\sqrt{x} + \frac{x\sqrt{x}}{3}\right) - \left(x^3 + \frac{x^3}{3}\right)\right] dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$= 2\frac{x^{7/2}}{7} + 2\frac{x^{5/2}}{3\times5} - \frac{x^4}{4} - \frac{x^4}{12}\Big]_0^1$$

$$= \left(\frac{2}{7} + \frac{2}{15} - \frac{1}{4} - \frac{1}{12}\right) - (0) = \frac{34}{420} = \frac{17}{210}$$

(j) Evaluate : $\int_0^2\int_1^2\int_0^{yz} xyz\,dx\,dy\,dz$ 8||

$$\Rightarrow \int_0^2\int_1^2 yz\left[\int_0^{yz} x\,dx\right]dy\,dz.$$

$$\Rightarrow \int_0^2\int_1^2 yz\left(\frac{x^2}{2}\right)_0^{yz} dy\,dz$$

$$\Rightarrow \frac{1}{2}\int_0^2\int_1^2 yz\left[y^2z^2-0\right]dy\,dz$$

$$\Rightarrow \frac{1}{2}\int_0^2\int_1^2 y^3z^3\,dy\,dz.$$

$$\Rightarrow \frac{1}{2}\int_0^2 z^3\left[\int_1^2 y^3\,dy\right]dz$$

$$\rightarrow \frac{1}{2\times4}\int_0^2 z^3\left[y^4\right]_1^2 dz$$

$$\rightarrow \frac{1}{8}\int_0^2 z^3\left[(2)^4-(1)^4\right]dz.$$

$$\rightarrow \frac{1}{8}\int_0^2 z^3\left[16-1\right]dz$$

$$= \frac{15}{8}\int_0^2 z^3\,dz$$

$$= \frac{15}{8}\times\left[\frac{z^4}{4}\right]_0^2 = \frac{15}{32}\times\left[16-0\right]$$

$$= \frac{15}{2}\ \text{Ans.}$$

Section-B

Q2 Prove that the sequence $\{a_n\}$ where a_n is $a_n = 1 + \frac{1}{2} + \frac{1}{3} - - - \frac{1}{n} - \log n$ is Convergent.

Sol:

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \text{---} + \frac{1}{n} - \log n.$$

$$a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \text{---} + \frac{1}{n} + \frac{1}{n+1} - \log(n+1).$$

$$\boxed{a_{n+1} - a_n} \to \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right).$$

$$= \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) \qquad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \frac{1}{n+1} - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots\right)$$

$$= \frac{1}{n+1} - \frac{1}{n} + \frac{1}{2n^2} - \left(\frac{1}{3n^3} - \frac{1}{4n^4}\right) - \cdots$$

$$< \frac{1}{n+1} - \frac{1}{n} + \frac{1}{2n^2} \qquad \left[\frac{1}{3n^3} - \frac{1}{4n^4} > 0\right].$$

$$= \frac{\cancel{2n^2} - \cancel{2n^2} - 2n + n + 1}{2n^2(n+1)}$$

$$= -\left[\frac{n-1}{2n^2(n+1)}\right]$$

$$= \leq 0 \quad \forall n \in N$$

$\boxed{a_{n+1} - a_n < 0}$ $\forall n \in N$.

$a_{n+1} < a_n \to \{a_n\}$ Monotonically $\downarrow$ing

also $a_n \geq 0$ $\forall n \in N$.

$a_n > 0$

$\therefore$ $\{a_n\}$ = bdd below.

$\therefore$ $a_n \Rightarrow$ Convgt.

Discuss the convergence or divergence of the series
$\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$.

Solution Here $a_n = \frac{1}{n} \sin \frac{1}{n}$.

$$a_n = \frac{1}{n}\left[\frac{1}{n} - \left(\frac{1}{n}\right)^3 \cdot \frac{1}{3!} + \left(\frac{1}{n}\right)^5 \cdot \frac{1}{5!} - \cdots \infty\right]$$

$$\left(\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \infty\right)$$

$$= \left(\frac{1}{n}\right)^2 - \left(\frac{1}{n}\right)^4 \cdot \frac{1}{3!} + \left(\frac{1}{n}\right)^6 \frac{1}{5!} - \cdots \infty$$

Take $b_n = \frac{1}{n^2}$

$$\therefore \frac{a_n}{b_n} = 1 - \left(\frac{1}{n}\right)^2 \cdot \frac{1}{3!} + \left(\frac{1}{n}\right)^4 \cdot \frac{1}{5!} - \cdots \infty.$$

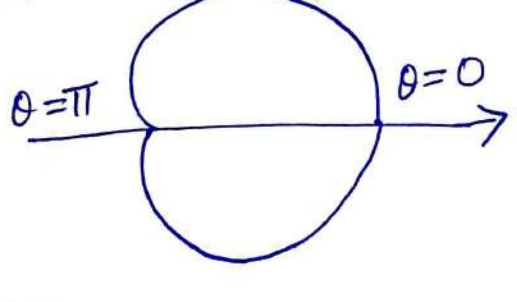
$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1 \quad \text{(finite \& non-zero)}$$

$\therefore$ By comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together.

We know that $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-Test.

$\therefore$ $\sum a_n = \sum \frac{1}{n} \sin \frac{1}{n}$ also converges.

Q-4 The curve $r = a(1+\cos\theta)$ revolves about the initial line. Find the volume of the figure formed.



$r = a(1+\cos\theta)$ is the equation of cardioid. It is symmterical about the initial line and for upper half, θ varies from 0 to 2π.

$$V = \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta.$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta.$$

$$= \frac{2}{3}\pi a^3 \int_0^{\pi} (1+\cos\theta)^3 \cdot \sin\theta \, d\theta.$$

Put $\cos\theta = u$ $\quad$ $\theta = 0 \ u = 1$

$-\sin\theta \, d\theta = du.$ $\quad$ $\theta = \pi \ u = -1$

$$= -\frac{2}{3}\pi a^3 \int_1^{-1} (1+u)^3 \, du$$

$$= \frac{2}{3}\pi a^3 \int_{-1}^{1} (1+u)^3 \, du = \frac{2}{3}\pi a^3 \left[\frac{(1+u)^4}{4}\right]_{-1}^{1}.$$

$$= \frac{2}{3}\pi a^3 \left[\frac{2^4}{4} - 0\right] = \frac{8}{3}\pi a^3.$$

Prove that $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx; \ m,n>0$ ①

Sol We know that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \text{— ①}$$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \text{— ⑪}$$

[∵ Splitting the limits of integration]

$$\beta(m,n) = I_1 + I_2 \quad \text{— ⑪}$$

where $I_2 = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

taking $x = \frac{1}{t}$

$\therefore \quad dx = -\frac{1}{t^2} dt$

Now $x = 1 \Rightarrow t = 1$ and $x \to \infty \Rightarrow t \to 0$

$$\therefore \quad I_2 = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\right) dt$$

$$I_2 = -\int_1^0 \frac{\frac{1}{t^{m-1}}}{\left(\frac{t+1}{t}\right)^{m+n}} \cdot \left(\frac{1}{t^2}\right) dt$$

$$I_2 = -\int_1^0 \frac{1}{t^{m-1}} \times \frac{t^{m+n}}{(t+1)^{m+n}} \times \frac{1}{t^2} dt$$

$$I_2 = -\int_1^0 \frac{t^{-(m-1)} \cdot t^{m+n} \cdot t^{-2}}{(t+1)^{m+n}} dt$$

$$I_2 = -\int_1^0 \frac{t^{-\cancel{m}+1+\cancel{m}+n-2}}{(t+1)^{m+n}} dt$$

$$I_2 = -\int_1^0 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$I_2 = \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt \quad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$ [$\because$ variable of integration can be changed in the definite integration

put I_1, I_2 in eq. (III), we get

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

Q6 If $U = x^3+y^3+x^3+3xyz$, show that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 3u$.

Sol Given $U = x^3+y^3+x^3+3xyz$

$$U = 2x^3+y^3+3xyz \quad \text{— (I)}$$

partial derivative of u with respect to x is

$$\frac{\partial U}{\partial x} = 6x^2+3yz \quad \text{— (II)}$$

partial derivative of u with respect to y is

$$\frac{\partial U}{\partial y} = 3y^2+3xz \quad \text{— (III)}$$

partial derivative of U with respect to z is

$$\frac{\partial U}{\partial z} = 3xy \quad \text{— (IV)}$$

Multiplying eq. (II) with x, eq. (III) with y and eq. (IV) with z, we get

$$x\frac{\partial U}{\partial x} = x(6x^2+3yz)$$

$$x\frac{\partial U}{\partial x} = 6x^3+3xyz \quad \text{— (V)}$$

$$y\frac{\partial u}{\partial y} = y(3y^2 + 3xz)$$

$$y\frac{\partial u}{\partial y} = 3y^3 + 3xyz \quad \text{— (VI)}$$

$$z\frac{\partial u}{\partial z} = z(3xy)$$

$$z\frac{\partial u}{\partial z} = 3xyz \quad \text{— (VII)}$$

adding eq. (V), (VI) and (VIII), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6x^3 + 3xyz + 3y^3 + 3xyz + 3xyz$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3(2x^3 + xyz + y^3 + xyz + xyz)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3(2x^3 + y^3 + 3xyz)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u \quad [\because \text{From eq.(1)}]$$

Obtain Taylor's expansion for $f(x,y) = y^x$ at $(1,1)$ upto second degree term.

Sol^n $f(x,y) = 1,1$

To find the expansion we use the following formula

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)\right] +$$

$$\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \cdots \quad -(1)$$

where (a,b) is the point at which we have to expand the function.

Here $(a,b) = (1,1)$

So we now find out the Partial derivatives

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y^x) = y^x \log y \Rightarrow \text{at } (1,1)$$

$$f_x(1,1) = 0 \quad -(2)$$

(as $\log 1 = 0$)

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^x) = xy^{x-1} \Rightarrow f_y(1,1) = 1 \quad -(3)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(xy^{x-1}) = x(x-1)y^{x-2} \Rightarrow f_{yy}(1,1) = 0 \quad -(4)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(y^x \log y) = y^x(\log y)^2 \Rightarrow f_{xx}(1,1) = 0 \quad -(5)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x\partial y} \text{ or } \frac{\partial^2 f}{\partial y\partial x} = y^{x-1} + y^x(\log y)^2$$

$$\Rightarrow f_{yy}(1,1) = 1 \quad -(6)$$

Substitute (2), (3), (4), (5) & (6) in equation (1)

We get

$$f(x,y) = 1 + (x-1)\cdot 0 + (y-1)\cdot 1 + (x-1)^2\cdot 0 + (y-1)^2\cdot$$
$$+ 2(x-1)(y-1)\cdot 1 + \cdots$$

$$= 1 + (y-1) + 2(x-1)(y-1) + \cdots$$

08) Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ by change of order of integration.

Sol^n :-> Clearly in the question y varies from x^2 to $2-x$.

we get $y = x^2$ to $y = 2-x$.

and x varies from 0 to 1

we get $x=0$ to $x=1$

We formulate the graph given in the adjacent figure. (Fig 1)

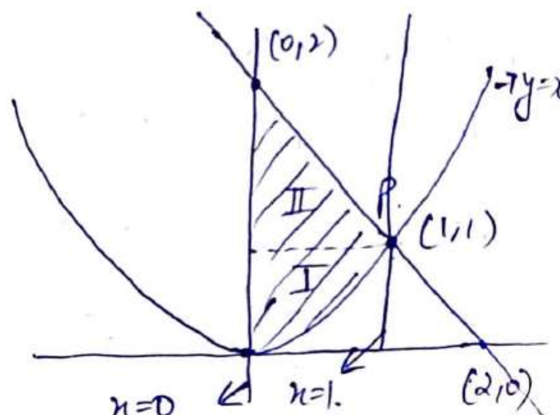


Fig 1

$y = 2-x$.

1) Find the point of intersection of the curve $y=x^2$ and $y=2-x$.

$$x^2 = 2-x \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x-1)(x+2) = 0$$
$$\Rightarrow x = 1, -2$$

So we get $P(1,1)$

We split the shaded region in two parts and then do the multiple integration.

Now limits become easy to calculate and given below

For I ~~$y=x^2$~~ $x = 0$ to $\sqrt{y}$
$y = 0$ to 1

For II $x = 0$ to $2-y$
$y = 1$ to 2

$$I = \int_0^1 \int_0^{\sqrt{y}} xy \, dx dy + \int_1^2 \int_0^{2-y} xy \, dx dy$$

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 y\left[(2-y)^2\right] dy$$

$$= \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 y\,(4+y^2-4y)\, dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2} \int_1^2 (4y+y^3-4y^2)\, dy$$

$$= \frac{1}{2}\left[\frac{1}{3}\right] + \frac{1}{2}\left[\frac{\cancel{4}y^2}{\cancel{2}} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[8+4-\frac{32}{3} - \left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[12-\frac{32}{3} - \left(\frac{24+3-16}{12}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[12-\frac{32}{3} - \frac{11}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{144-128-11}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2} \times \frac{5}{12} = \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24} = \frac{9}{24}$$

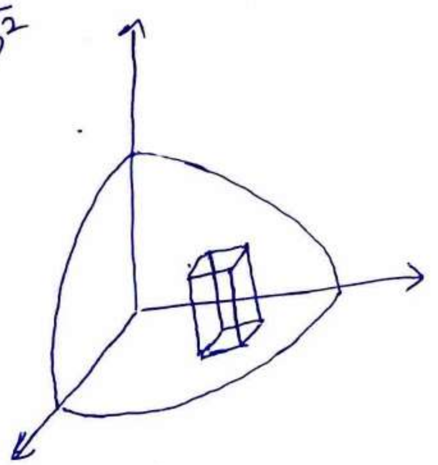
Q. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Soln. The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Since ellipsoid is symmetrical about the axis.

$\therefore$ volume of ellipsoid is 8 times the volume in the first octant. Now in the xy-plane, $z=0$ so the portion of ellipsoid becomes $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$, which is an ellipse. Here x vary from 0 to a and y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

In the positive octant z is $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

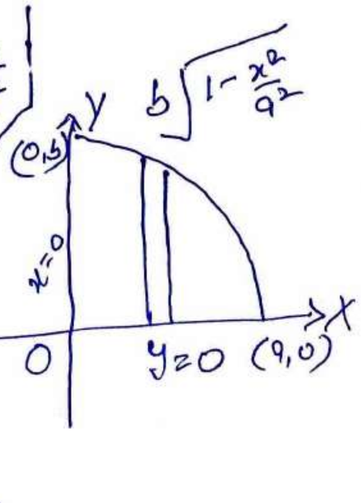


Hence the volume of the ellipsoid

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} z \, dy\, dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

$$= 8\int_0^a \int_0^t c\sqrt{\frac{t^2}{b^2}-\frac{y^2}{b^2}}\, dy\, dx, \text{ where } \sqrt{1-\frac{x^2}{a^2}}=\frac{t}{b}$$



$$= 8\int_0^a \int_0^t \frac{c}{b}\sqrt{t^2-y^2}\, dy\, dx$$

$$= 8\int_0^a \frac{c}{b}\left[\frac{y\sqrt{t^2-y^2}}{2}+\frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t dt$$

$$= \frac{8c}{b}\int_0^a \left[\left(0+\frac{t^2}{2}\sin^{-1}(1)\right)-0\right]dx$$

$$= \frac{2\pi c}{b}\int_0^a t^2\,dx = \frac{2\pi c}{b}\int_0^a b^2\left[1-\frac{x^2}{a^2}\right]dx$$

$$= 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a = 2\pi bc\left(a-\frac{a}{3}\right) = \frac{4}{3}\pi abc.$$

Ans.